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Approximate solutions for fuzzy Volterra integro-differential equations

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Abstract: We introduce here a simple efficient Haar Wavelet Method for numerical solution of a class of Fuzzy Volterra integro-differential equations of the second kind. The present technique depends on converting nonlinear Fuzzy Volterra integro-differential equations into to system of algebraic equations which solved by using a suitable numerical method. Numerical examples are given to delineate efficiency and accuracy of the present technique. Comparison of the results is gotten by the Haar Wavelet Method with the exact solution.

Keywords: Haar Wavelet, fuzzy Volterra integro-differential equations, product matrix, coefficient matrix, operational matrix.

1 Introduction

The significance of fuzzy integro-differential equations (FIDEs) have stimulated a large amount of research work in recent years. Nonlinear phenomena that appear in many applications in mathematics, physical and engineering can be modeled by FIDEs. Many experts in such areas extensively use FIDEs to make some problems under study more understandable. Usually, data about these problems involved is pervaded with uncertainty, which can arise in the experiment part, data collection, measurement process as well as the initial values are determined. Classical mathematics cannot cope with this situation. Therefore, it is necessary to have some mathematical apparatus in order to understand this uncertainty [1]. Besides, fuzzy calculus theory is a powerful tool for modeling uncertainty and processing vague of subjective information in mathematical models. Their main directions of development have been diverse and its applications to the very varied real problems, including the golden mean, particle systems, quantum optics and chaotic system [2,6]. The reader is asked to refer to [7,10] to know more details about integral equations.

The numerical solvability of fuzzy initial value problems (FIVPs) have discussed recently by many authors [11,12] but, generally, investigation about a fuzzy IDEs is scarce, especially, discussion on a nonlinear case. However, FIDEs are almost impossible to solve as well as do not always have solutions by using analytical or semi-analytical techniques. Therefore, it is necessary to develop new numerical techniques to approximate the solution for the FIVPs. In this paper, we extend the application of the Haar Wavelet Method to provide symbolic approximate solution for fuzzy Volterra integro-differential equations.

The structure of this paper is organized as follows; In section 2, some preliminaries of fuzzy integral equation are presented. In section 3, the FVIDE are discussed. In section 4, Analysis of Haar Havelet Method then we use it for finding the numerical solution of FVIDE. In section 5, the proposed methods are implemented for solving two illustrative examples. Finally, conclusion is provided in section 6.

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2 Preliminaries of Fuzzy Integral Equation

In this section we give some preliminaries and notations used in fuzzy calculus. We start by defining the fuzzy number.

Definition 1. *A fuzzy number is a fuzzy set* $y : R^1 \to E^1 = [0,1]$ *which satisfying the following properties.*

(i) y is upper semi continuous on R,
(ii) The support $\overline{\{u \in R|\, y(u) > 0\}}$is a compact set,
(iii) $y$is a fuzzy convex set (i.e., $y(\lambda u + (1-\lambda)v) \geq \min\{y(u), y(v)\} \forall u, v \in R, \lambda \in [0,1]$),
(iv) There are real numbers $a$and b,$c \leq a \leq b \leq d$, for which
(a) $y(u)$is monotonically increasing on $[c,a]$,
(b) $y(u)$is monotonically decreasing on $[b,d]$,
(c) $y(u) = 1$for $a \leq u \leq b$.
The set of all the fuzzy numbers is denoted by E^1 is given by Kaleva [13].

Definition 2. *A fuzzy number y is represented by an ordered pair of* $(\underline{y}(r), \bar{y}(r))$ *of functions* $\underline{y}(r)$*and* $\bar{y}(r)$, $0 \leq r \leq 1$ *which satisfying the following conditions.*

1. $\underline{y}(r)$ is a bounded monotonically increasing, left continuous function on (0,1] and right continuous at 0,
2. $\bar{y}(r)$ is a bounded monotonic decreasing, left continuous function on (0,1] and right continuous at 0,
3. $\underline{u}(r) \leq \bar{u}(r)$,$0 \leq$ r ≤ 1.

A crips number r is simply represented by $\underline{u}(\alpha)$=$\bar{y}(\alpha)$=r,0 $\leq \alpha \leq$ 1. This fuzzy number space can be embedded into the Banach space $B = \bar{C}[0,1] \times \bar{C}[0,1]$.
For arbitrary $y(r)$=($\underline{y}(r)$,$\bar{y}(r)$), $x(r)$=($\underline{x}(r)$,$\bar{x}(r)$), and $k > 0$ we define addition $(x+y)$and scalar multiplication by k as,

1.$\underline{(x+y)}(r)$=$\underline{x}(r)$+$\underline{y}(r)$,
2.$\overline{(x+y)}(r)$ =$\bar{x}(r)$+$\bar{y}(r)$,
3.$\underline{(kx)}$ (r)= $k\bar{x}(r)$,$\overline{(kx)}(r) = k\underline{x}(r)$ if $k < 0$,
4.$\underline{(kx)}$ (r)=$k\underline{x}(r)$,$\overline{(kx)}(r) = k\bar{x}(r)$ if $k \geq 0$.

Definition 3. *Let* $f : [a,b] \to E^1$ *be a bounded function. Then the function*

$$\varphi_{[a,b]}(f, \delta) = \sup\{D(f(x), f(y)); \quad x, y \in [a,b], |x-y| \leq \delta, \}$$

is called the modulus of continuity of f on $[a,b]$.

Definition 4. *Let* $F : [a,b] \to E^1$. *For each partition* $P = \{t_0, t_1, \ldots, t_n\}$ *of* $[a,b]$ *and for arbitrary* $\xi_i \in [t_{i-1}, t_i]$, $1 \leq i \leq n$ *suppose,*

$$R_P = \sum_{i=1}^{n} F(\xi_i)(t_i - t_{i-1}), \qquad \Delta := \{|t_i - t_{i-1}|, \quad i = 1, \ldots, n\}.$$

The definite integral $F(t)$*over* $[a,b]$ *is* $\int_a^b F(t)dt = \lim\limits_{\Delta \to 0} R_P$.

If the fuzzy function$F(t)$is continuous function in the metric D, its definite integral exists. Also,

$$\overline{\left(\int_a^b F(t;\alpha)dt\right)} = \int_a^b \overline{F}(t;\alpha)dt, \qquad \underline{\left(\int_a^b F(t;\alpha)dt\right)} = \int_a^b \underline{F}(t;\alpha)dt,$$

where $\underline{F}(t,r), \bar{F}(t,r)$ is the parametric form of $F(t)$. It should be noted that the fuzzy integral can be also defined by using the Lebesgue-type approach [14]. However, if $F(t)$is continuous, both approaches yield the same value.

Lemma 1. *[15] If f and g are Henstock integrable functions and if the function given by $D(f(t),g(t))$ is Lebesgue integrable, then,*

$$D\left((FH)\int_a^b f(t)dt,(FH)\int_a^b g(t)dt\right) \leq (L)\int_a^b D(f(t),g(t))dt.$$

Definition 5. *[16] A function $F:(a,b)\to E^1$ is called H-differentiable at $t\in(a,b)$ if, for $h>0$ sufficiently small, there exist the H-differences $F(t+h)-F(t)$, $\quad F(t)-F(t-h)$ and an element $F'(t)\in E^1$ such that*

$$\lim_{h\to 0^+} D\left(\frac{F(t+h)-F(t)}{h},F'(t)\right) = \lim_{h\to 0^+} D\left(\frac{F(t+h)-F(t)}{h},F'(t)\right)=0.$$

Then $F'(t)$ is called the fuzzy derivative of F at t.

Definition 6. *[17] Let I be a real interval. A mapping $\bar{v}:I\to E$ is called a fuzzy process and we denote the $r-$ set by $[y(t)]_r=[\underline{y}(t,r),\bar{y}(t,r)]$. The seikkala derivative $\tilde{y}^{/}$ of $\tilde{y}$ is defined by $[y^{/}(t)]_r=[\underline{y^{/}}(t,r),\bar{y}^{/}(t,r)]$, provided that is an equation defines a fuzzy number $\tilde{y}^{/}(t)\in E$.*

Definition 7. *[17] The fuzzy integral of a fuzzy process $\tilde{y}$, $\int_a^b y(t)\,dt$ for $a,b\in I$ and I be a real interval is defined by $[\int_a^b y(t)\,dt]_r=[\int_a^b \underline{y}(t,r)\,dt,\int_a^b \bar{y}(t,r)\,dt]$, provided that the Lebesgue integrals on the right exist.*

Theorem 1. *(i) The pair $(E,+)$ is a commutative semi group with $\tilde{0}=\chi_{\{0\}}$ zero element,*
(ii) At fuzzy number which are not crisp, there is no opposite element (that is, $(E,+)$ cannot be a group)
(iii) At any $\lambda\in R$ with $a,b\geq 0$ or $a,b\leq 0$, and for any $x\in E$, we have $(a+b)\times x=(a\times x+b\times x)$,
(iv) At any $\lambda\in R$ and $x,y\in E$ we have $\lambda\times(x+y)=\lambda\times x+\lambda\times y$,
(v) At any $\lambda,\mu\in R$ and $u\in E$ we have $\lambda\times(\mu\times x)=(\lambda\cdot\mu)\times x$,
(vi) The function $\|.\|_E:E\to R_+\bigcup\{0\}$ has the usual properties of the norm, so $\|x\|_E=0$ if $u=\tilde{0}$, $\|\lambda\times x\|_E=|\lambda|\,\|x\|_E+\|y\|_E$ for any $x,y\in E$,
(vii) $|\|x\|_E-\|y\|_E|\leq D(x,y)$ and $D(x,y)\leq\ |\|x\|_E+\|y\|_E|$ for any $x,y\in E$.
Since $(E,+)$ is not a group, but only a commutative monoid, the structure $(E,+,\times,\|.\|_E)$ is not normed space.

3 Fuzzy Volterra integro-differential equations

We consider the VIDE of the form

$$y'(x)=f(x,y(x))+\lambda\int_0^x k(x,t)y(t)dt, y(t_0)=y_0, \tag{1}$$

where $\lambda>0$, $k(t,s)$ is an arbitrary kernel function and $f(x,y(x))+\lambda\int_0^x k(x,t)y(t)dt$ is a continuous fuzzy function. We assume that $\lambda>0$. In order to design a numerical scheme for solving equation (1) we write the parametric form of the given equation (1) as follows

$$\underline{y}'(x,r)=\underline{f}(x,\underline{y}(x,r),\bar{y}(x,r))+\lambda\int_0^x \underline{U}(t,r)dt, \tag{2}$$

$$\bar{y}'(x;r)=\overline{f}(x,\underline{y}(x,r),\bar{y}(x;r))+\lambda\int_0^x \overline{U}(t,r)dt,$$

where

$$\underline{U}(t,r)=\underline{k(t,s)y(s;\alpha)}=\begin{cases} k(x,t)\underline{y}(t,r), & k(x,t)\geq 0\\ k(x,t)\bar{y}(t,r), & k(x,t)<0,\end{cases}$$

and

$$\overline{U}(t,r)=\overline{k(x,t)y(s;r)}=\begin{cases}k(x,t)\overline{y}(t,r), & k(x,t)\geq 0\\ k(x,t)\underline{y}(t,r), & k(x,t)<0.\end{cases} \tag{3}$$

We can see that Eq (2) is a system of FVIDE for each $0\leq\alpha\leq 1$ and $a\leq t\leq b$.

4 Haar Wavelet Method

In this section, The Haar functions are an orthogonal family of switched rectangular waveforms where amplitudes can differ from one function to another.

Definition 8. *The Haar Wavelet family is defined over the interval [0,1] as follows*

$$h_i(x)=\begin{cases}1 & \alpha\leq x<\beta\\ -1 & \beta\leq x<\gamma\\ 0, & elsewhere.\end{cases} \tag{4}$$

Where $\alpha=\frac{k}{m}$, $\beta=\frac{k+0.5}{m}$, and $\gamma=\frac{k+1}{m}$. The integer $m=2^j, j=0,1,2,\ldots,J$, indicates the level of the wavelet, and $k=0,1,2,\ldots,m-1$ is the translation parameter. Maximal level of resolution is the integer J. The index i is calculated according to the formula $i=m+k+1$. In case of minimal values $m=1$, $k=0$. we have $i=2$; The maximal value of i in is $i=2M=2^{J+1}$. It is assumed that for value $i=1$, the corresponding scaling function in $[0,1]$ is as

$$h_1(x)=\begin{cases}1, & 0\leq x\leq 1,\\ 0, & elsewhere.\end{cases} \tag{5}$$

Let us define the collocation points $x_l=\frac{l-0.5}{2M}$, where $l=1,2,\ldots,2M$ and discretize the Haar function $h_i(x)$. The Haar coefficient matrix is defined as $HAAR(i,l)=h_i(x_l)$ which is a square matrix of order $2M\times 2M$.

4.1 Method of solution

We can assume that $H\in L^2[0,1)$ is a Hilbert space with the inner product defined by $\langle f(x),h\rangle=\int_0^1 \mathrm{f}(x)\,\mathrm{h}^T(x)\,dx$ since V is a finite-dimensional bspace of H, it is closed and convex. Thus, for $w\in H$,there are a unique approximation out of V such as g,

$$\|w-g\|\leq\|g-w\|,\forall h\in V.$$

Since $g\in V$, there exist a unique coefficient $c_1,c_2,....,c_{2M}$ such that

$$f(x)=\sum_{i=0}^{2M}c_i\,h_i(x), \tag{6}$$

where the Haar Wavelet Method coefficients are given by $c_i=\frac{(f(x),h_{(\mu)}(x))}{(h_{(\mu)}(x),h_{(\mu)}(x))}$such that $(.,.)$denotes the inner product. where $\mu=2M=2^{J+1}$, $C$and h(x) are $1\times 2M$ matrices given by

$$C=[c_1,c_2,c_3,\ldots\ldots,c_{2M}],h_{(2M)}(x)=[h_1(x)\quad h_2(x)\quad\ldots\quad h_{2M}(x)]. \tag{7}$$

We can also approximate the function $k(x,t) \in L^2[0,\,1) \times [0,\,1)$ as follows

$$k(x,t) \cong h^T{}_{(2M)}(x)\; K\, h_{(2M)}(x), \tag{8}$$

where K is an $2M \times 2M$ matrix that we can obtain as

$$K = \frac{(h_{(2M)}(x), (k(x,t), h_{(2M)}(t))}{(h_{(2M)}(x), h_{(2M)}(x))(h_{(2M)}(t), h_{(2M)}(t)).} \tag{9}$$

4.2 Operational matrix of integration

The vector integration $h_{(\mu)}(\tau)$ is given by

$$\int_0^x h_{(\mu)}(\tau)\; d\tau \approx P_{(\mu\times\mu)}\; h_{(\mu)}(x). \tag{10}$$

Where $P_{(\mu\times\mu)}$ is the $2M \times 2M$ operational matrix for integration. Carrying out these integrations with the aid of (10) we find

$$P_{i,1}(x) = \begin{cases} x-\alpha & x\varepsilon[\alpha,\beta) \\ \gamma - x, & x\varepsilon[\beta,\gamma) \\ o, & elsewhere, \end{cases} \tag{11}$$

$$P_{i,2}(x) = \begin{cases} \frac{1}{2}(x-\alpha)^2, & x\varepsilon[\alpha,\beta) \\ \frac{1}{4m^2} - \frac{1}{2}(\gamma-x)^2, & x\varepsilon[\beta,\gamma) \\ \frac{1}{4m^2}, & x\varepsilon[\gamma,1) \\ o, & elsewhere, \end{cases} \tag{12}$$

$$P_{i,3}(x) = \begin{cases} \frac{1}{6}(x-\alpha)^3, & x,\varepsilon[\alpha,\beta) \\ \frac{1}{4m^2}(x-\beta) - \frac{1}{6}(\gamma-x)^3, & x\varepsilon[\beta,\gamma) \\ \frac{1}{4m^2}(x-\beta), & x\varepsilon[\gamma,1) \\ o, & elsewhere\;, \end{cases} \tag{13}$$

$$P_{i,4}(x) = \begin{cases} \frac{1}{24}(x-\alpha)^4, & x\varepsilon[\alpha,\beta) \\ \frac{1}{8m^2}(x-\beta)^2 - \frac{1}{24}(\gamma-x)^4 + \frac{1}{192m^4}, & x\varepsilon[\beta,\gamma), \\ \frac{1}{8m^2}(x-\beta)^2 + \frac{1}{192m^4} & x\varepsilon[\gamma,1) \\ o, & elsewhere. \end{cases} \tag{14}$$

The Heaviside step function $F(x)$ is defined as

$$f(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

By using Heaviside step function, we can also write

$$h_n(x) = F\left(x - \frac{k}{2^j}\right) - 2F\left(x - \frac{k+0.5}{2^j}\right) + F\left(x - \frac{k+1}{2^j},\right), \tag{15}$$

$$n = 2^j + k, j, k \in N \cup \{0\},\;\; 0 \leq k \leq 2^j.$$

4.3 The integration of the cross product

The integration of the cross product of two Haar Wavelets vectors can be obtained as

$$D=\int_0^1 h_{(i)}(\tau)\ h^T{}_{(l)}(\tau)\,d(\tau)=\begin{cases}\frac{1}{m}, & i=l\\ 0, & i\neq l.\end{cases} \tag{16}$$

We discretize the functions $h_{(i)}(x)$ by dividing the interval $x\in[0,1]$ into $2M$ parts of equal length $\Delta x=\frac{1}{2M}$ and introduce the collocation points

$$x_i=\frac{i-\frac{1}{2}}{2M},\quad i=1,2,\ldots,2M.$$

4.4 Multiplication of Haar Wavelet Method

It is always necessary to evaluate the product of $h_{(\mu)}(x)$ and $h^T{}_{(\mu)}(x)$, that is called the product matrix of Haar Wavelet Method. Let

$$M(x)\cong h_{(\mu)}(x)\ h^T{}_{(\mu)}(x,)$$

, where $M(x)$ is $2M\times 2M$ matrix. Multiplying the matrix $M(x)$by vector C we obtain

$$M(x)C=\tilde{C}h_{(\mu)}(x), \tag{17}$$

where $\tilde{C}$ is $2M\times 2M$ matrix and called the coefficient matrix. Let $R$is $2M\times 2M$ matrix. Multiplying the matrix $R$by vector $h_{(\mu)}(x)$ and multiplying the matrix $h_{(\mu)}(x)$ by the resulted matrix $R\ \ h_{(\mu)}(x)$, we obtain

$$h^T_{(\mu)}(x)\ R\,h_{(\mu)}(x)=\tilde{R}h_{(\mu)}(x). \tag{18}$$

where $\tilde{R}$ is $1\times 2M$ matrix and called the coefficient matrix With the powerful properties of (17). We can achieve $\tilde{R}$by away like $\tilde{C}$ we can convert the Volterra part of integral and Integro-Differential equations System equations to an algebraic equation.

4.5 Error analysis

Let $y(x)$ is the exact solution and $y_m(x)$ is the approximate solution of fuzzy Volterra integro-differential equations. $ERROR_m(x)$ be the corresponding error function and is defined as,

$$ERROR_m(x)=|y(x)-y_m(x)|\cong\sum_{j=J+1}^{\infty}\sum_{k=0}^{2^j-1}c_{2^j+k}\,h_{2^j+k}(x),\ x\ \in\ [0,\ 1].$$

Theorem 2. *Suppose that $y(x)$ satisfies a Lipschitz's condition on $[0,\ 1]$, then there exists constant $K\ >\ 0$ (dependent on both $y(x)$ and interval), such that*

$$|y(x1)\ -\ y(x2)|\ \leq\ K|x1\ -\ x2|,\ \forall x1,\ x2\ \in\ [0,\ 1].$$

The Haar Wavelet Method will be convergent in the sense that $ERROR_m(x)$ goes to zero as m goes to infinity. The order of convergence is

$$\|ERROR_m(x)\|_2 = 0(\frac{1}{m}).$$

Proof. Squaring the integrand and breaking the summation

$$\|ERROR_m(x)\|_2{}^2 = \int_0^1 \Big(\sum_{j=J+1}^{\infty}\sum_{k=0}^{2^j-1} c_{2^j+k}\, h_{2^j+k}(x)\Big)^2 dx,$$

we obtain

$$\|ERROR_m(x)\|_2{}^2 = \sum_{j=J+1}^{\infty}\sum_{k=0}^{2^j-1} c^2{}_{2^j+k}\int_0^1 h^2{}_{2^j+k}(x)\,dx$$
$$+\sum_{j=J+1}^{\infty}\sum_{k=0}^{2^j-1}\sum_{p=J+1}^{\infty}\sum_{q=0,\ q\neq k}^{2^p-1} c_{2^j+k}\, c_{2^p+q}\int_0^1 c_{2^j+k}c_{2^p+q}(x)dx.$$

Using orthogonality properties, we obtain

$$\|ERROR_m(x)\|_2{}^2 = \sum_{j=J+1}^{\infty}\sum_{k=0}^{2^j-1} c^2{}_{2^j+k}\left(\frac{1}{2^j}\right),$$

for $i = 2^j + k$, we obtain

$$c_{2^j+k} = 2^j\int_0^1 y(x)h_{2^j+k}(x)\,dx.$$

From mean value theorem, there exist two points described below

$$x_{11}^{jk}\in\left(\frac{k}{2^j},\frac{k+0.5}{2^j}\right), x_{22}^{jk}\in\left(\frac{k+0.5}{2^j},\frac{k+1}{2^j}\right),$$

so

$$c_{2^j+k} = 2^j[(\left(\frac{k+0.5}{2^j}-\frac{k}{2^j}\right)y\left(x_{11}^{jk}\right)+\left(\frac{k+1}{2^j}-\frac{k+0.5}{2^j}\right)y(x_{22}^{jk}),$$

using Lipschitz's condition, we obtain

$$c_{2^j+k} = \frac{1}{2}[y\left(x_{11}^{jk}\right)-y(x_{22}^{jk})] \le \frac{1}{2}k(x_{11}^{jk}-x_{22}^{jk})] \le (\frac{1}{2})k\left(\frac{1}{2^j}\right)\cong k(\frac{1}{2^{j+1}}\Big),$$

so

$$\|ERROR_m(x)\|_2{}^2 = \sum_{j=J+1}^{\infty}\sum_{k=0}^{2^j-1} k^2\frac{1}{2^{2J+2}}(\frac{1}{2^{2J}}) = \frac{k^2}{4}\sum_{j=J+1}^{\infty}2^J\left(\frac{1}{2^{3J}}\right),$$

we obtain

$$\|ERROR_m(x)\|_2{}^2 = \frac{k^2}{3}(\frac{1}{2^J+1})^2.$$

But, $m = 2^J+1$, therefore, we obtain

$$\|ERROR_m(x)\|_2 = 0(\frac{1}{m}).$$

Thus, when m goes to infinity, then $ERROR_m(x)$ tends to zero.

4.6 Solving Fuzzy Volterra integro-differential equations by Haar Wavelet Method

In this section, we introduce an approximation method (Haar Wavelet Method) to solve the Volterra integro-differential equations in the form

$$\begin{aligned} &y'(x) = f(x,y(x)) + \lambda \textstyle\int_0^x k(x,t)y(t)dt, \quad a \le x \le b, \\ &y(x_0) = y_0, \end{aligned} \tag{19}$$

where $f(x,y(x)) \in L^2([0,1))$, $k(\mathrm{x,t}) \in L^2([0,1)) \times L^2([0,1))$ with $a=0$, $b=1$ and$\lambda > 0$.
$(0 \le r \le 1)$ is an arbitrary kernel function and $y(\mathrm{x})$ is the unknown fuzzy real valued function.
We write the parametric form of the given fuzzy integral equations system as follows

$$\underline{y}'(x,r) = \underline{f}(x,r) + \lambda \int_0^x k(x,t)\,\underline{y}(t,r)dt,$$

$$\bar{y}'(x,r) = \bar{f}(x,r) + \lambda \int_0^x k(x,t)\bar{y}(t,r)dt, \tag{20}$$

where

$$k(x,t)\,\underline{y}(t,r) = \begin{cases} k(x,t)\,(\underline{y}(t,r)) & k(x,t) \ge 0 \\ k(x,t)\,(\bar{y}(t,r)) & k(x,t) < 0, \end{cases}$$

and

$$k(x,t)\,\bar{y}(t,r) = \begin{cases} k(x,t)\,(\bar{y}(t,r)\,) & k(x,t) \ge 0 \\ k(x,t)(\underline{y}(t,r)) & k(x,t) < 0. \end{cases} \tag{21}$$

We can approximate the function $\underline{y}(x,r)$, $\bar{y}(x,r)$, $\underline{f}(x,r)$, $\bar{f}(x,r)$ and $k(x,t)$by Haar Wavelet Method as follows

$$\underline{y}(x,r) \approx h^T_{(\mu)}(x)\,\underline{Y_1}\,h_{(\mu)}(r)\,, \quad \bar{y}(x,r) \approx h^T_{(\mu)}(x)\,\overline{Y_2}\,h_{(\mu)}(r)\,,$$

$$\underline{f}(x,r) \approx h^T_{(\mu)}(x)\,\underline{F_1}\,h_{(\mu)}(r)\,, \quad \bar{f}(x,r) \approx h^T_{(\mu)}(x)\,\overline{F_2}\,h_{(\mu)}(r)\,, \tag{22}$$

$$k(x,t) \approx h^T_{(\mu)}(x)\,K\,h_{(\mu)}(t)\,, \quad \underline{y^{/}}(x,r) \approx h_{(\mu)}(r)\;\underline{Y_1^{/}}^{T}\,h_{(\mu)}(x)\,,$$

$$\overline{y^{/}}(x,r) \approx h_{(\mu)}(r)\,\overline{Y^{/T}}\,h_{(\mu)}(x)\,.$$

Which $\underline{y^{/}}(x,r)$ and $\overline{y^{/}}(x,r)$will be evaluated in terms $\underline{y}(x,r)$ and $\bar{y}(x,r)$

$$\underline{y}(x,r) \;=\; \int_0^x \underline{y^{/}}(t,r)\,dt + \underline{y}(0).$$

If we expand $\underline{y}(0)$with Haar Wavelet Method basis i.e. $\underline{y}(0) = \underline{Y_0}\,h_{(\mu)}(x)$ then $\underline{Y_0}$ is obtained as

$$\underline{Y_0} \;=\; \left[\underbrace{\overbrace{\underline{y}(0), \underline{y}(0), \ldots, \underline{y}(0),} \;\; \ldots\ldots, \; \overbrace{\underline{y}(0), \underline{y}(0), \ldots, \underline{y}(0),}}_{2M} \right] \tag{23}$$

$$\underline{Y_1^T}\,h_{(\mu)}(x) \;\cong\; \int_0^x \underline{Y_1^{/}}^{T}\,h_{(\mu)}(t)\,dt + \underline{Y_0^T}\,h_{(\mu)}(x)\,,$$

$$\cong \underline{Y_1^{/}}^{T} \int_0^x h_{(\mu)}(t)\,dt + \underline{Y_0^T}\,h_{(\mu)}(x)\,,$$

$$\cong \underline{Y_1}^{/T} P_{(\mu\times\mu)}\ h_{(\mu)}(x) + \underline{Y_0}^T h_{(\mu)}(x),$$

$$\cong (\underline{Y_1}^{/T} P_{(\mu\times\mu)} + \underline{Y_0}^T)\, h_{(\mu)}(x),$$

and we have

$$\underline{Y_1}^T \cong \underline{Y_1}^{/T} P_{(\mu\times\mu)} + \underline{Y_0}^T. \tag{24}$$

Therefore,

$$\underline{Y_1}^{/} \cong (p^T_{(\mu\times\mu)})^{-1}(\underline{Y_1} - \underline{Y_0}). \tag{25}$$

After substituting the approximate equations (22), (28) into equations (20) and (21), we get

$$\begin{aligned} h^T_{(\mu)}(x)\,(p^T_{(\mu\times\mu)})^{-1}(\underline{Y_1} - \underline{Y_0})\,h_{(\mu)}(r) &= h^T_{(\mu)}(x)\,\underline{F_1}\,h_{(\mu)}(r) \\ &+\lambda \textstyle\int_0^x h^T_{(\mu)}(x)\,K\,h_{(\mu)}(t)\ h^T_{(\mu)}(t)\ \underline{Y_1}\,h_{(\mu)}(r)\,dt\ , \\ h^T_{(\mu)}(x)\,(p^T_{(\mu\times\mu)})^{-1}(\overline{Y_2} - \overline{Y_0})\,h_{(\mu)}(r) &= h^T_{(\mu)}(x)\,\overline{F_2}\,h_{(\mu)}(r) \\ &+\lambda \textstyle\int_0^x h^T_{(\mu)}(x)\,K\,h_{(\mu)}(t)\ h^T_{(\mu)}(t)\ \overline{Y_2}\,h_{(\mu)}(r)\,dt. \end{aligned} \tag{26}$$

We have

$$\begin{aligned} h^T_{(\mu)}(x)\,(p^T_{(\mu\times\mu)})^{-1}(\underline{Y_1} - \underline{Y_0})h_{(\mu)}(r) &= h^T_{(\mu)}(x)\,\underline{F_1}\,h_{(\mu)}(r) \\ &+\lambda\, h^T_{(\mu)}(x)\,K \textstyle\int_0^x h_{(\mu)}(t)\ h^T_{(\mu)}(t)\,dt\ \underline{Y_1}\,h_{(\mu)}(r)\ , \\ h^T_{(\mu)}(x)\,(p^T_{(\mu\times\mu)})^{-1}(\overline{Y_2} - \overline{Y_0})\,h_{(\mu)}(r) &= h^T_{(\mu)}(x)\,\overline{F_2}\,h_{(\mu)}(r) \\ &+\lambda\, h^T_{(\mu)}(x)\,K \textstyle\int_0^x h_{(\mu)}(t)\ h^T_{(\mu)}(t)\,dt\ \overline{Y_2}\,h_{(\mu)}(r), \end{aligned}$$

with the powerful properties of equation (10) we get

$$\begin{aligned} h^T_{(\mu)}(x)\,(p^T_{(\mu\times\mu)})^{-1}(\underline{Y_1} - \underline{Y_0})h_{(\mu)}(r) &= h^T_{(\mu)}(x)\,\underline{F_1}\,h_{(\mu)}(r) \\ &+\lambda\, h^T_{(\mu)}(x)\,K\,\underline{\tilde{Y}_1}\,P_{(\mu\times\mu)}\,h_{(\mu)}(x)\ h_{(\mu)}(r)\ , \\ h^T_{(\mu)}(x)\,(p^T_{(\mu\times\mu)})^{-1}(\overline{Y_2} - \overline{Y_0})\,h_{(\mu)}(r) &= h^T_{(\mu)}(x)\,\overline{F_2}\,h_{(\mu)}(r) \\ &+\lambda\, h^T_{(\mu)}(x)\,K\,\overline{\tilde{Y}_2}\,P_{(\mu\times\mu)}\,h_{(\mu)}(x)\ h_{(\mu)}(r). \end{aligned} \tag{27}$$

Therefore

$$(p^T_{(\mu\times\mu)})^{-1}(\underline{Y_1} - \underline{Y_0}) = \underline{F_1}\,h_{(\mu)}(r) + \lambda\, h^T_{(\mu)}(x)\,K\,\underline{\tilde{Y}_1}\,P_{(\mu\times\mu)}\,h_{(\mu)}(x)\ h_{(\mu)}(r), \tag{28}$$

$$(p^T_{(\mu\times\mu)})^{-1}(\overline{Y_2} - \overline{Y_0}) = \overline{F_2}\,h_{(\mu)}(r) + \lambda\, h^T_{(\mu)}(x)\,K\,\overline{\tilde{Y}_2}\,P_{(\mu\times\mu)}\,h_{(\mu)}(x)\ h_{(\mu)}(r). \tag{29}$$

Where, the dimensional subscripts have been dropped to simplify the notation. Rewriting (29), we have

$$\underline{Y_1} = p^T_{(\mu\times\mu)}\,(\underline{F_1}\,h_{(\mu)}(r) + \lambda\, h^T_{(\mu)}(x)\,K\,\underline{\tilde{Y}_1}\,P_{(\mu\times\mu)}\,h_{(\mu)}(x)\ h_{(\mu)}(r)) + \underline{Y_0},$$

$$\overline{Y_2} = p^T_{(\mu\times\mu)}\,(\,\overline{F_2}\,h_{(\mu)}(r) + \lambda\, h^T_{(\mu)}(x)\,K\,\overline{\tilde{Y}_2}\,P_{(\mu\times\mu)}\,h_{(\mu)}(x)\ h_{(\mu)}(r)\,) + \overline{Y_0}. \tag{30}$$

From (30) we have a system of $2M\times 2M$ nonlinear equations and $2M$ unknowns. After solving above nonlinear system using Newton method. We can achieve the unknown vectors $\underline{Y_1}$ and $\overline{Y_2}$. The required approximated solutions. $\underline{y}(x,r) \approx$

$h^T_{(\mu)}(x)\,\underline{Y_1}\,h_{(\mu)}(r)$, $\bar{y}(x,r) \approx h^T_{(\mu)}(x)\,\overline{Y_2}\,h_{(\mu)}(r)$ for the nonlinear Fuzzy Fredholm integro-differential Equations (20) and (21).

5 Numerical examples

In this section, we give the approximate solution of linear and nonlinear FVIDE by using the Haar Wavelet Method.

Example 1. Consider the following linear FVIDE

$$\begin{array}{l} \underline{y}'(t,r) = (0.5+0.5r)(e^t - t) + \int_0^t xt\,\underline{y}(x,r)dx \\ \bar{y}'(t,r) = (2-r)(e^t - t) + \int_0^t xt\,\bar{y}(x,r)dx, \\ \underline{y}(0) = 0.5+0.5r,\ \bar{y}(0) = 2-r, \quad 0 \le r \le 1, \quad 0 \le x \le t, \quad t \in [0,1]. \end{array} \tag{31}$$

The exact solution is given by $\underline{y}(t,r) = (0.5+0.5r)e^t, \qquad \bar{y}(t,r) = (2-r)e^t.$

The exact and obtained approximate solutions of FVIDE are compared in Table 1.

Table 1: Table to test captions and labels

x	Exact solution	Haar Wavelet Method solution at $2M=8$	Absolute error at $2M=8$	Absolute error at $2M=16$
0.1	0.9983341665	0.9983341428	2.37×10^{-8}	4.61×10^{-14}
0.2	0.9933466540	0.9933467756	1.260×10^{-7}	7.02×10^{-14}
0.3	0.9850673556	0.9850674729	1.173×10^{-8}	2.35×10^{-14}
0.4	0.9735458558	0.9735458342	2.161×10^{-8}	5.11×10^{-14}
0.5	0.9588510772	0.9588503494	7.278×10^{-8}	2.73×10^{-13}
0.6	0.9410707892	0.9410707712	1.80×10^{-8}	6.35×10^{-13}
0.7	0.9203109820	0.9203110906	1.086×10^{-7}	1.05×10^{-11}
0.8	0.8966951136	0.8966952102	9.66×10^{-8}	3.24×10^{-11}
0.9	0.8703632328	0.8703632216	1.12×10^{-8}	4.01×10^{-11}

Example 2. We consider the following linear FVIDE

$$\begin{array}{l} \underline{y}'(x,r) = (r-1) + \int_0^x \underline{y}(t,r)dt, \\ \bar{y}'(x,r) = (1-r) + \int_0^x \bar{y}(t,r)dt, \\ \underline{y}(0) = 0;\ \bar{y}(0) = 0; \quad 0 \le r \le 1, \quad 0 \le t \le x. \end{array} \tag{32}$$

The exact solution is given by $\underline{y}(x,r) = (r-1)\sinh(x);\ \bar{y}(x,r) = (1-r)\sinh(x).$

Table 2 compare the maximum absolute errors of Haar Wavelet Method at different values of M. These results have been included to demonstrate the validity and capability of Haar Wavelet Method, for a certain value of M increases, the accuracy increases and the numerical results obtained for this example are highly agreed with the exact results.

Table 2: Error of $\underline{y}(x,r)$ and $\overline{y}(x,r)$ of Example 2

x	Absolute error of $\underline{y}(x,r)$using Haar Wavelet Method at $2M=8$	Absolute error of $\overline{y}(x,r)$using Haar Wavelet Method at $2M=8$	Absolute error of $\underline{y}(x,r)$using Haar wavelet method at $2M=16$	Absolute error of $\overline{y}(x,r)$using Haar Wavelet Method at $2M=16$
0.1	4.205×10^{-6}	2.876×10^{-6}	1.11×10^{-9}	1.02×10^{-9}
0.2	5.305×10^{-6}	1.836×10^{-6}	3.15×10^{-9}	4.19×10^{-9}
0.3	2.07×10^{-6}	1.295×10^{-6}	1.04×10^{-9}	3.01×10^{-9}
0.4	3.11×10^{-6}	8.303×10^{-6}	2.07×10^{-9}	2.09×10^{-9}
0.5	1.320×10^{-5}	5.442×10^{-5}	7.44×10^{-9}	1.05×10^{-8}
0.6	4.351×10^{-5}	9.314×10^{-5}	1.36×10^{-9}	4.01×10^{-8}
0.7	4.130×10^{-5}	8.874×10^{-5}	6.23×10^{-9}	3.04×10^{-9}
0.8	1.025×10^{-5}	5.862×10^{-5}	6.53×10^{-8}	4.03×10^{-8}
0.9	9.04×10^{-5}	4.542×10^{-5}	3.22×10^{-8}	2.05×10^{-8}

6 Conclusion

We used an appropriate technique for finding the approximate solution of FVIDE. The results show that the resolved method is a promising tool for this type of fuzzy integral equations. In this paper, a new technique based on the Haar Wavelet Method was developed. For this aim, we compute operational matrices and apply them for solving FVIDE. These polynomials have some advantage to other polynomials, for example these polynomials are orthonormal. Thus, coefficient of expansion can be calculated easily. By applying these matrices and collocation method FVIDE convert to linear systems of algebraic equations which can be solved by a suitable numerical method. Also, the errors of the present method were investigated. In section 5, we solved two examples by proposed technique that numerical result is performed on computer by using a program written in Maple.

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